ISRAEL JOURNAL OF MATHEMATICS 100 (1997), 125–161

INTRINSIC ERGODICITY OF SMOOTH INTERVAL MAPS

BY

JÉRÔME BUZZI

Université Paris-Sud, Département de Mathématiques
91405 Orsay, France
e-mail: jerome.buzzi@math.u-psud.fr

ABSTRACT

We generalize the technique of Markov Extension, introduced by F. Hofbauer [10] for piecewise monotonic maps, to arbitrary smooth interval maps. We also use A. M. Blokh's [1] Spectral Decomposition, and a strengthened version of Y. Yomdin's [23] and S. E. Newhouse's [14] results on differentiable mappings and local entropy.

In this way, we reduce the study of C^r interval maps to the consideration of a finite number of irreducible topological Markov chains, after discarding a small entropy set. For example, we show that C^∞ maps have the same properties, with respect to intrinsic ergodicity, as have piecewise monotonic maps.

1. Introduction

We consider the measurable dynamical system defined by a C^r self-map f of the interval. Following the "intrinsic" point of view introduced by B. Weiss [22], we do not distinguish *a priori* one invariant measure but we are interested in all (ergodic) invariant, probability measures with large entropy. Particularly we ask:

> How large is $\mathrm{Max}(f)$, the set of **maximal measures**, i.e., ergodic, invariant, probability measures which maximize metric entropy? What conditions do imply **intrinsic ergodicity**, that is, existence of a unique maximal measure?

F. Hofbauer [10] has studied in this respect piecewise monotonic maps, i.e., interval maps with **finite** critical and discontinuity sets. He has shown that, for

Received August 28, 1995

these maps, Max(f) is never empty and is finite as soon as the topological entropy of f is positive. Here we show that these results remain when the finiteness of the critical set is replaced by a smoothness assumption. E.g., for a C^∞ map, whose critical set can be uncountable, we have:

THEOREM 1.1: *Let f: $[0,1] \to [0,1]$ be C^∞. If the topological entropy of f is positive ($h_{\text{top}}(f) > 0$), then:*

(1) Max(f) *is non-empty and finite,*
(2) *topological transitivity implies intrinsic ergodicity.*

In section 2 below we state the results of this paper. We then proceed to the proofs. In section 3, we deduce from Y. Yomdin's theory on differentiable mappings [23] a bound on "local entropy", which seems of general interest. It allows us to get a simple proof of Newhouse's existence result. In section 4, we use this bound to study the topological entropy in A.M. Blokh's Spectral Decomposition [1]. In this way, we prove a topological multiplicity theorem about "transitive components".

In section 5, we study the structure of "transitive components", so that the previous multiplicity result gives the result above. We do it by generalizing methods of F. Hofbauer [10]: we represent the smooth system by building its "Markov extension", i.e., a topological Markov chain, and prove an isomorphism theorem.

An important feature of this generalization is that it can be applied to some non-trivial classes of *multi-dimensional* dynamical systems [5].

ACKNOWLEDGEMENT: I am indebted to Philippe Thieullen for many discussions which have greatly improved this paper, as well as for introducing me to this subject in the first place.

2. Statement of results

2.1 LOCAL ENTROPY. We recall some definitions (due to R. Bowen [2]). Consider f: $X \to X$ a continuous self-map on some compact, metric space. The (ϵ, n)-**ball** centered at some point x is:

$$B_n(x,\epsilon) = \{y \in X \colon d(f^k(y), f^k(x)) < \epsilon \ \forall k = 0, \dots, n-1\}.$$

An (ϵ, n)-**cover** of $S \subset X$ is a subset $R \subset X$ such that $S \subset \bigcup_{x \in R} B_n(x, \epsilon)$.

Finally, $r(\epsilon, n, Y)$ is the minimum cardinal of a (ϵ, n)-cover of some (not necessarily invariant) subset Y. We recall that the topological entropy of f, $h_{\text{top}}(f)$, can be computed as the limit, when δ does to zero, of:

$$h_{\text{top}}(f, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log r(\delta, n, X).$$

We can now give:

Definition 2.1: The **local entropy** of f, $h_{\text{loc}}(f)$, is defined to be the limit when $\epsilon \to 0+$ of:

$$h_{\text{loc}}(f, \epsilon) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X} \log r(\delta, n, B_n(x, \epsilon)).$$

We prove in section 3:

THEOREM 2.2: *Let M be a compact, riemannian, m-dimensional manifold with boundary and $f: M \to M$ a C^r self-map with $1 \leq r \leq \infty$. Let $M(f) = \max_{x \in M} \|T_x f\|_x$ and $R(f) = \inf_{k \geq 1} \sqrt[k]{M(f^k)}$ (spectral radius of Tf). Then:*

$$h_{\text{loc}}(f) \leq \frac{m}{r} \log R(f).$$

Then we remark that, from this, it is easy to deduce the following result due to S.E. Newhouse, avoiding the use of Pesin theory:

THEOREM 2.3 (S.E. Newhouse [14]): *Let M be a compact, riemannian manifold with boundary and $f: M \to M$ a C^∞ map. Then:*

$$\text{Max}(f) \neq \emptyset.$$

Local entropy will also be crucial for the following results.

2.2 ENTROPY IN THE SPECTRAL DECOMPOSITION. A.M. Blokh's Spectral Decomposition (Theorem 4.4 below) implies that, for a continuous self-map of the interval, the non-wandering set (which in particular carries every invariant probability measure) is the union of the transitive components, defined as follows:

Definition 2.4: Let $f: X \to X$ be a self-map on some topological space. A **transitive subset** is a subset T of X such that

(i) T is f-invariant: $f(T) \subset T$,
(ii) the restriction $f: T \to T$ is topologically transitive: for every open set U, V intersecting T, there exists $n \geq 1$, such that $f^{-n}(U) \cap V \neq \emptyset$.

A **transitive component** is a subset $T \subset X$ such that:

(i) T is a transitive subset,

(ii) if $T' \subset X$ is another transitive subset then either $T' \cap T$ is finite or $T' \subset T$.

MULTIPLICITY THEOREM 2.5: *Let f: $[0,1] \to [0,1]$ be C^r with $1 < r \leq \infty$. Let $M(f) = \sup_{x \in [0,1]} |f'(x)|$ and $R(f) = \inf_{k \geq 1} \sqrt[k]{M(f^k)}$. Assume that H is a constant such that:*

$$\frac{1}{r} \log R(f) < H \leq h_{\text{top}}(f).$$

Then there are a finite, non-zero number of transitive components T with topological entropy $h_{\text{top}}(f|T) \geq H$.

Taking $H = h_{\text{top}}(f)$ we solve the topological analogue to intrinsic ergodicity:

COROLLARY 2.6: *Let f: $[0,1] \to [0,1]$ be C^r with $1 < r \leq \infty$. If:*

$$h_{\text{top}}(f) > \frac{1}{r} \log R(f) \tag{2.1}$$

then the number of transitive components T with maximum entropy, i.e. such that:

$$h_{\text{top}}(f|T) = h_{\text{top}}(f),$$

is finite and non-zero.

Remark: The bound $\log R(f)/r$ is sharp as shown by the examples in Appendix A.

2.3 COMBINATORIAL MODEL. We now turn to the description of the structure inside a given transitive component. The following notion will give us a combinatorial model of the measurable dynamics of these components:

Recall that every **oriented graph** G (countable, maybe finite, set with some arbitrary relationship $\to$) defines a **Topological Markov Chain**, or **T.M.C.** Take the action of the left shift σ on the set $\Sigma(G)$ of two-sided infinite paths on G:

$$\Sigma(G) = \left\{(a_n)_{n \in \mathbb{Z}} \in G^{\mathbb{Z}}: a_n \to a_{n+1} \text{ is an arrow of } G\right\},$$
$$\sigma((a_n)_{n \in \mathbb{Z}}) = (a_{n+1})_{n \in \mathbb{Z}}.$$

G also defines a one-sided T.M.C. $(\Sigma_+(G), \sigma)$ in an obvious way.

G is said to be **irreducible** if for every $(a, b) \in G^2$ there exists a path from a to b, i.e., a finite sequence $g_0, \ldots, g_n$ such that $g_0 = a$, $g_n = b$ and $g_i \to g_{i+1}$ for $i = 0, \ldots, n-1$.

The following quantity is chosen as the "intrinsic size" of subsets:

Definition 2.7: Let $f: X \to X$ be a measurable self-map. We denote by $\mathcal{M}_f(X)$ the set of f-invariant probability measures, by $\mathcal{M}_f^{\text{erg}}(X) \subset \mathcal{M}_f(X)$ the subset of ergodic, invariant, probability measures, and by $h_\mu(f)$ the usual metric entropy of (X, f, μ), for $\mu \in \mathcal{M}_f(X)$.

Let Y be a measurable subset of X, not necessarily invariant. We define the **metric entropy of the subset** Y with respect to f to be:

$$h_{\text{met}}(f, Y) = \sup\{h_\mu(f) \colon \mu \in \mathcal{M}_f^{\text{erg}}(X) \text{ such that } \mu(Y) > 0\}$$

(if there is no such μ we set $h_{\text{met}}(f, Y) = 0$). The metric entropy of the whole of X is simply $h(f) = h_{\text{met}}(f, X) = \sup\{h_\mu(f) \colon \mu \in \mathcal{M}_f(X)\}$.

Remarks: Y is a null-set for every invariant, ergodic, probability measure with entropy strictly greater than $h_{\text{met}}(f, Y)$.

If X is compact and f continuous, $h(f)$ is the topological entropy, $h_{\text{top}}(f)$ (according to the "variational principle", see, e.g., [6]).

Once we adopt this definition of size, we get the following notion of isomorphism (see, for instance, S.E. Newhouse and L.-S. Young [16] for similar ideas):

Definition 2.8: Let $f: X \to X$ and $g: Y \to Y$ be two measurable self-maps. We will say that (X, f) and (Y, g) are h-**conjugated** if there are forward invariant subsets $X' \subset X$, $Y' \subset Y$ such that:

(i) there exists $\psi : X' \to Y'$, bi-measurable (ψ is measurable and ψ^{-1} is well-defined and also measurable) and such that $\psi \circ f = g \circ \psi$ (we say that ψ is a **measurable isomorphism** between (X', f) and (Y', g)).

(ii) The neglected parts are small:

$$\max(h_{\text{met}}(f, X \setminus X'), h_{\text{met}}(g, Y \setminus Y')) < h(f).$$

The left hand side of the above inequality is called the **constant** of the h-isomorphism.

The inequality (ii) above is obviously equivalent to the ones obtained by substituting any of $h(g)$, $h_{\text{met}}(f, X')$ or $h_{\text{met}}(g, Y')$ for the right hand side $h(f)$.

Recall that the **natural extension** $\bar{f}: \bar{T} \to \bar{T}$ of $f: T \to T$ is defined as follows:

$$\bar{T} = \{\bar{x} \in T^{\mathbb{Z}}: \forall n \in \mathbb{Z} \quad \bar{x}_{n-1} \in f^{-1}(\bar{x}_n)\},$$
$$\bar{f}(\bar{x}) = \bar{y}, \qquad \bar{y}_n = \bar{x}_{n+1} \quad (n \in \mathbb{Z}).$$

The map $\bar{\pi}: \bar{T} \to T$ defined by $\bar{\pi}(\bar{x}) = \bar{x}_0$ makes $(\bar{T}, \bar{f})$ a topological extension of (T, f). More importantly, it is well-known (e.g., [17, p. 13]) that the induced map between sets of invariant probability measures $\bar{\pi}_*: \mathcal{M}_{\bar{f}}(\bar{T}) \to \mathcal{M}_f(T)$ is onto and one-to-one and preserves entropy and ergodicity: $h_{\bar{\pi}_*\bar{\mu}}(f) = h_{\bar{\mu}}(\bar{f})$ and $\bar{\mu}$ is ergodic iff $\bar{\pi}_*\bar{\mu}$ is.

We are at last in position to state the:

STRUCTURE THEOREM 2.9: *Let $f: [0,1] \to [0,1]$ be C^r with $1 < r \leq \infty$. Let $M(f) = \sup_{x \in [0,1]} |f'(x)|$ and $R(f) = \inf_{k \geq 1} \sqrt[k]{M(f^k)}$. Let T be a transitive component with topological entropy $h_{\text{top}}(f|T) > \frac{1}{r} \log R(f)$.*

The dynamics on T are described by an irreducible topological Markov chain in the sense that the natural extension of (T, f) is h-conjugated to the chain.

2.4 APPLICATION TO INTRINSIC ERGODICITY. The structure theorem together with the results of B.M. Gurevič [9] (Theorem 7.1 below) allows us to deduce from the Multiplicity Theorem the following **uniqueness** result:

COROLLARY 2.10: *Let f, r and $R(f)$ be as above. If the topological entropy of f is big enough, i.e., if:*

$$h_{\text{top}}(f) > \frac{1}{r} \log R(f),$$

then the number of maximal measures is finite and their natural extensions are Markov measures on a T.M.C.: for instance they are Bernoulli.

If, in addition, f is topologically transitive, then the maximal measure, if it exists, is unique.

2.5 QUESTIONS. Theorem 2.5 completely solves the question of finite/infinite topological multiplicity under smoothness assumptions. One can ask for a more precise result, like in the piecewise monotonic case (see [4]):

• Is there an explicit and "reasonable" bound (i.e., given by a simple formula and not too far from being sharp). The difficulty is that Yomdin's theory does not give such estimates.

On the one hand, Newhouse's result solves the existence problem for arbitrary compact C^∞ dynamical systems. On the other hand, we have seen that there exists C^r interval maps, with r arbitrarily large but finite with *no* maximum entropy transitive component. More interestingly, in Appendix A, we build simple examples of C^r interval maps which are topologically transitive and have no maximal measure. We remark that they have small entropy: $h_{\text{top}}(f) < \frac{1}{r}\log R(f)$. So we ask:

- Does the condition $h_{\text{top}}(f) > \frac{1}{r}\log R(f)$ imply existence of a maximal measure?

The main remaining questions are about the dynamics *inside* a transitive component or, more simply, for a topologically transitive f:

- Is this condition necessary to ensure uniqueness? to ensure h-isomorphism to a T.M.C.?
- More generally, is the condition $h_{\text{top}}(f) > \frac{1}{r}\log R(f)$ really relevant for the dynamic *inside* transitive components of C^r interval maps? i.e., do *new phenomena* appear when this condition is weakened (e.g., replaced by f $C^{1+\alpha}$ and $h_{\text{top}}(f) > 0$)?

3. Local entropy

Let us make some remarks about our definition of local entropy 2.1 above. It is inspired by R. Bowen's definition of asymptotic h-expansivity [3] (see also S.E. Newhouse's definition [14]). Notice the place of the "$\sup_{x\in X}$": our definition provides a uniform bound and thus it majorizes these previous, finer notions, therefore it will be much easier to use. We will use it directly in the next section.

Before proving Theorem 2.2, we remark that this result implies a similar result proved by S.E. Newhouse for his local entropy using Pesin theory (because it was based on Y. Yomdin explicit *volume* estimates). As in Newhouse's paper [14] we get:

COROLLARY 3.1 (S.E. Newhouse [14]): *Let $f\colon M \to M$, $R(f)$ be as above. Then $\frac{m}{r}\log R(f)$ bounds the defect in upper-semicontinuity of the metric entropy with respect to the vague topology.*

In particular, if f is C^∞, then the metric entropy is upper-semicontinuous. Therefore:

$$\operatorname{Max}(f) \neq \emptyset.$$

Indeed, it is clear by standard methods (contained in M. Misiurewicz's proof of the variational principle [13]) that local entropy bounds the defect in upper-semicontinuity of the metric entropy. We have:

PROPOSITION 3.2 (after S.E. Newhouse [14]): *Let $f\colon X \to X$ be a continuous map on a compact metric space.*

Assume that μ_n is some sequence in $\mathcal{M}_f(X)$ converging to some μ. Then:

$$\limsup_{n\to\infty} h_{\mu_n}(f) \le h_\mu(f) + h_{\mathrm{loc}}(f).$$

(For the sake of completeness we give a proof in Appendix B.)

The proof of Theorem 2.2 is modelled after the proof of Yomdin's result [23, Proposition 2.2] about "local volume", $v^0_{m,r}(f)$, which is the growth rate of the volume, counted with multiplicities, of the image of $B_n(x,\epsilon)$ by f^n as $n \to \infty$, for small $\epsilon > 0$. The main point of the proof below, in comparison with Yomdin's proof, is the remark that the reparametrizations in Y. Yomdin's theory can be assumed to be contracting ($d(\psi(x),\psi(y)) \le d(x,y)$) and "telescoping" (point (3) of the claim below).

For simplicity, we pretend that M is simply an open subset of $\mathbb{R}^m$ and refer to [23] for the general case.

The first step of the proof is the strengthening of Y. Yomdin's Theorem 2.1 in [23]:

All statements made above are valid with the differentiability order r of f not necessarily integer. We will say that f is C^r with $0 < r < \infty$ if: (i) f is k times continuously differentiable (with $k = \lfloor r \rfloor$ the largest integer with $k < r$); (ii) $f^{(k)}$ is Hölder with exponent $r - k \in (0,1]$: there exists a constant $K < \infty$ s.t. $|f^{(k)}(x) - f^{(k)}(y)| \le K|x-y|^{r-k}$. The smallest such constant is called the **Hölder constant** of order r of f and is denoted by $\mathrm{Höl}_r(h)$.

We say that a map h is C^r ($r > 1$) on the *closed* cube $Q^m = [0,1]^m$ if it is C^r on the interior of Q^m and all differentials of order up to r can be continuously extended to the border of Q^m. We write $R = \lfloor r \rfloor$, $\alpha = r - \lfloor r \rfloor \in (0,1]$. The **$C^r$-size** of h, $\|h\|_r$, is:

$$\|h\|_r = \sup\{\mathrm{Höl}_r(h), \|d^s h(x)\|\colon s = 1,\dots,R \text{ and } x \in \operatorname{int} Q^m\}.$$

($\|d^s h(x)\|$ is the maximum of the absolute values of the partial derivatives of order s of all components of h at x.)

PROPOSITION 3.3: *Let $f\colon B(0,2) \subset \mathbb{R}^m \to \mathbb{R}^m$ be C^r. Let $\sigma\colon Q^m \to B(0,2)$ be C^r with $\|\sigma\|_r \leq 1$.*

Then there exists $\psi_1, \dots, \psi_q$ with $q \leq C_1 \max(\|f\|_r, 1)^{m/r}$ such that

(1) *$\psi_i : Q^m \to Q^m$ is a C^r mapping,*
(2) *$\bigcup_{i=1}^q \mathrm{Im}(\psi_i) \supset (f \circ \sigma)^{-1}(B(0,1))$,*
(3) *$\|d^s(f \circ \sigma \circ \psi_i)\|_r \leq 1$,*
(4) *every ψ_i is* **contracting**.

The constant C_1 depends only on m and r.

To prove this, it is enough to apply Y. Yomdin's Theorem 2.1 to the "complete" mapping $x \mapsto (x, f \circ \sigma(x))$. For the sake of completeness, we give a self-contained proof of the one-dimensional case. It is considerably simpler than the general case and is the only one we shall need.

Proof of the proposition for $m = 1$: Let $g = f \circ \sigma$. A simple computation shows that $\|g\|_r \leq K\|f\|_r$, with K a constant, depending only on r. Subdividing $Q^1 = [0,1]$ into $\lceil (100K\|f\|_r)^{1/r} \rceil$ subintervals of equal length and using the corresponding affine reparametrizations we are reduced to as many mappings $h\colon [0,1] \to \mathbb{R}$ such that h is C^r and:

$$\mathrm{H\ddot{o}l}_r(h) \leq 1/100.$$

We fix one of the mappings h.

Let H be the polynomial defined by the Taylor developement of degree R of h at $1/2$. We see that $|h^{(s)} - H^{(s)}| \leq 1/100$ on $[0,1]$, for $s = 1, \dots, R$.

Where $|H| > 1 + 1/100$, $|h| > 1$: the image falls outside the ball of interest $B(0,1)$. We can restrict ourselves to the subintervals J of $[0,1]$ defined by $|H| \leq 1 + 1/100$. There are at most R of them. Take the corresponding affine reparametrizations $\phi\colon [0,1] \to J$. For each one, $H \circ \phi$ is a polynomial of degree R with $|H \circ \phi| \leq 1 + 1/100$ on $[0,1]$. But, on the finite-dimensional vector space defined by the polynomials of degree at most R, all norms are equivalent:

$$\max_{\substack{x\in[0,1]\\ 0\leq s\leq R}} |(H \circ \phi)^{(s)}(x)| \leq K' \max_{x\in[0,1]} |(H \circ \phi)(x)| \leq K'(1 + 1/100)$$

with K' depending only on R.

It is therefore enough to subdivide again into $\lceil K'(1 + 1/100) \rceil$ subintervals of equal length and use the related affine reparametrizations ψ to bound by 1 all derivatives of $h \circ \phi \circ \psi$ up to order R. We remark that all our reparametrizations are affine and contracting. Therefore $\text{Höl}_r(h \circ \phi \circ \psi) \leq \text{Höl}_r(h) \leq 1$ as well.

The proposition is proved with $\{\psi_1, \ldots, \psi_q\} = \{\phi \circ \psi\}$ and

$$C_1 = ((100K)^{1/r} + 1) \cdot R \cdot (K'(1 + 1/100) + 1). \qquad \blacksquare$$

Proof of the theorem: Let $x_0 \in M$. Fix $\epsilon_0 > 0$ so small that $2\epsilon_0 \|f\|_r \leq M(f)$ and $\epsilon_0 < 1/2$. Consider "charts" along the orbit of x_0:

$$\chi_k \colon x \in B(f^k(x_0), \epsilon_0) \mapsto t_0 + (x - f^k(x_0))/2\epsilon_0 \in B(t_0, 1/2) \subset Q^m$$

with t_0 the center of Q^m.

Define $f_k = \chi_{k+1} \circ f \circ \chi_k^{-1}$, $F_k = f_{k-1} \circ \cdots \circ f_0$ ($F_0 = \text{Id}$) and $V_n = \{y \in \mathbb{R}^m \colon F_k(y) \in B(t_0, 1/2) \quad \text{for } k = 0, \ldots, n-1\}$. Write $\kappa = C_1 M(f)^{m/r}$.

Remark that $\|f_k\|_r \leq M(f)$ and $\chi_0(B_n(x_0, \epsilon_0)) = V_n$. So we can study $B_n(x_0, \epsilon_0)$ by applying the previous proposition to the f_i's.

Claim: *There exists a family of C^r maps $\{\psi_{n,i} : Q^m \to M \colon n \geq 0, \quad 1 \leq i \leq \kappa^n\}$ such that, for all $n \geq 0$:*

(1) $\bigcup_{i=1}^{\kappa^n} \text{Im}(\psi_{n,i}) \supset V_{n+1}$,
(2) $\|F_n \circ \psi_{n,i}\|_r \leq 1$ *for* $i = 1, \ldots, \kappa^n$,
(3) *if* $n > 0$, *for every* $i \in \{1, \ldots, \kappa^n\}$ *there exists* $j \in \{1, \ldots, \kappa^{n-1}\}$ *and a* **contracting** *map* $\phi_{n,i}^{n-1,j}$ *such that:*

$$\psi_{n,i} = \psi_{n-1,j} \circ \phi_{n,i}^{n-1,j}$$

(we say j is defined by i and n),

(4) $\psi_{n,i}$ *is contracting for* $i = 1, \ldots, \kappa^n$.

We shall write $\phi_{n,i}^{m,j}$ *for* $\phi_{m+1,j(m+1)}^{m,j} \circ \cdots \circ \phi_{n,i}^{n-1,j(n-1)}$ *with* $j(k)$ *defined by* $j(k+1)$ *and* $k+1$ *as in* (3) *above.*

This is essentially [23, Lemma 2.3] with the addition of the telescoping and contracting properties (3) and (4). Using the previous proposition instead of Theorem 2.1 of [23], the necessary modifications of the proof are clear.

Let $\delta > 0$. We shall build a $(\delta, n+1)$-cover for $B_{n+1}(x_0, \epsilon_0)$ for all $n \geq 0$.

Select R a finite $(\delta, 1)$-covering of Q^m. Let:

$$S = \chi_0^{-1} \left(\bigcup_{i=1}^{\kappa^n} \psi_{n,i}(R) \right).$$

Let us show that S is an $(\delta, n+1)$-covering of $B_{n+1}(x_0, \epsilon_0)$. Let $x \in B_{n+1}(x_0, \epsilon_0)$. By claim (1):

$$\bigcup_i \operatorname{Im}(\chi_0^{-1} \circ \psi_{n,i}) \supset B_{n+1}(x_0, \epsilon_0),$$

so there exists $t \in Q^m$ and $i \in \{1, \ldots, \kappa^n\}$ such that:

$$x = \chi_0^{-1} \circ \psi_{n,i}(t).$$

Now, there exists $t' \in R$ such that $|t' - t| \leq \delta$. Write x' for the image of t' by $\chi_0^{-1} \circ \psi_{n,i}$. Let us prove that $x \in B_{n+1}(x', \delta)$. For every $m = 0, \ldots, n$:

$$f^m(x) = f^m \circ \chi_0^{-1} \circ \psi_{n,i}(t) = \chi_m^{-1} \circ F_m \circ \psi_{n,j}(t)$$

and likewise for $f^m(x')$. By claim (3) above:

$$\psi_{n,j} = \psi_{m,k} \circ \phi_{n,j}^{m,k} \qquad \text{for some } k.$$

But the mappings: $\phi_{n,j}^{m,k}$, $F_m \circ \psi_{m,k}$ and χ_m^{-1} are contracting. Therefore:

$$|f^m(x) - f^m(x')| \leq |t - t'| \leq \delta.$$

Hence, S is indeed a $(\delta, n+1)$-cover for $B_{n+1}(x_0, \epsilon_0)$. The cardinal of this cover is bounded by $\kappa^n \operatorname{card} R$, with R independent of n, so that:

$$h_{\text{loc}}(f) \leq h_{\text{loc}}(f, \epsilon_0) \leq \log \kappa = \frac{m}{r} \log M(f) + C$$

with $C = \log C_1$ a constant depending only on m and r. To get rid of C just write $h_{\text{loc}}(f) = h_{\text{loc}}(f^q)/q \leq \frac{m}{r} \log M(f) + C/q$ and let $q \to \infty$. ∎

4. Entropy in the spectral decomposition

Throughout this section f is a continuous self-map of the unit interval. For such maps, A.M. Blokh [1] has shown the existence of a "Spectral Decomposition" somewhat similar to Smale's Spectral Decomposition for Axiom-A diffeomorphisms [19]. This decomposition can be derived by the analysis of the periodic intervals, defined as follows:

Definition 4.1: An interval $J \subset [0,1]$ is said to be a **periodic interval** if it is compact, has non-empty interior and if there exists an integer $n \geq 1$ such that:

(i) $J, f(J), \ldots, f^{n-1}(J)$ have pairwise disjoint interiors.

(ii) $f^n(J) = J$.

n is called the **period** of J. The **orbit** of J is $\operatorname{orb}(J) = J \cup f(J) \cup \cdots \cup f^{n-1}(J)$.

It is sometimes convenient to consider the set $\operatorname{orb}(J)$ and not to have to choose one periodic interval. Let us remark that $\operatorname{orb}(J) = \operatorname{orb}(I)$ (I, J periodic intervals) does not imply that $I = f^k(J)$ for some $k \geq 0$.

Definition 4.2: A subset $C \subset [0,1]$ such that $C = \operatorname{orb}(J)$ for some periodic interval J is called a **cycle**.

The number of connected components of the cycle is called its **period**. Each connected component is said to be a periodic interval **defined** by the cycle C.

We write $\mathcal{C}$ for the set of cycles.

Clearly a periodic interval defined by a cycle is indeed a periodic interval with period equal to the period of the cycle.

Remark: Any non-wandering, non-periodic point belongs to some cycle. This accounts for the rôle of these cycles in building the Spectral Decomposition.

A. M. Blokh has introduced the following subsets:

Definition 4.3: (A.M. Blokh [1]). To every cycle $C \in \mathcal{C}$ is associated the following subset of $[0,1]$:

$$\operatorname{PEC}(C) = \left\{ x \in [0,1] \colon \forall \epsilon > 0 \ \overline{\bigcup_{k \geq 0} f^k(B(x,\epsilon) \cap C)} = C \right\}.$$

If $\operatorname{PEC}(C) \neq \emptyset$, we call it a **positive entropy component** (P.E.C. for short).

We can now quote (part of) A.M. Blokh's theorem:

THEOREM 4.4 (A.M. Blokh's Spectral Decomposition Theorem [1]): *Let $f \colon [0,1] \to [0,1]$ be continuous.*

Every transitive subset is included in a transitive component.

Every transitive component either has zero topological entropy or can be written as $\operatorname{PEC}(C)$ *for some cycle C.*

Reciprocally, every subset of the form $\operatorname{PEC}(C) \neq \emptyset$ *(with C a cycle) is a transitive component with positive topological entropy.*

For future use (section 7) we make the transitivity properties a little more precise. $\Omega(f)$ is the set of non-wandering points.

LEMMA 4.5: *Let $K = \mathrm{PEC}(C) \neq \emptyset$ with C a cycle. For every $x \in K \cap \Omega(f) \cap \operatorname{int} C$ and $\epsilon > 0$,*

$$\bigcup_{k\geq 0} f^k(B(x,\epsilon)\cap C) \supset C \setminus Y$$

with Y a finite set, depending only on K, not on x and ϵ.

We shall only prove —and use— that Y can be chosen a countable set.

Proof: Select some countable base $\mathcal{V}$ of the topology.

For every x and ϵ as above, take $V \in \mathcal{V}$ such that $V \subset B(x,\epsilon)$, $x \in V$. Let U be the connected component of the forward invariant set $\bigcup_{k\geq 0} f^k(V\cap C)$ containing x. As $x \in \operatorname{int} C$, U contains a neighbourhood of x. As x is non-wandering, there exists $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. But this implies $f^n(U) \subset U$. Therefore:

$$\bigcup_{k\geq 0} f^k(B(x,\epsilon)\cap C) \supset \bigcup_{k\geq 0} f^k(V\cap C) = \bigcup_{k=0}^{n-1} f^k(U)$$

is a finite union of intervals: taking the closure can add only finitely many points Y_1, depending on the open set V (and the smaller is V the larger is Y_1). So one may take Y to be the union of the Y_1's when the open set V ranges in the countable base $\mathcal{V}$. ∎

We remark that two cycles may overlap each other. This is the motivation behind the following definitions.

Definition 4.6: Let C be a cycle. The **excluded set** of C is:

$$X(C) = \bigcup_{c} \operatorname{int}(c),$$

where c ranges over the cycles which do not contain C. The **reduced cycle** C' of C is defined to be:

$$C' = \bigcup_{I} \overline{\operatorname{conv}(I \setminus X(C))}$$

where I ranges over the connected components of C and conv is the convex closure.

Remark that, except if $C' = C$, the reduced cycle C' is not a cycle: it is not even forward invariant.

LEMMA 4.7:

(i) *If C is a cycle and C' is its reduced cycle then:*

$$\mathrm{PEC}(C) \subset C' \subset C.$$

(ii) *Let $C_1, C_2 \in \mathcal{C}$. The two reduced cycles C_1', C_2' have interiors either disjoint or nested. If the reduced cycles are nested, then the true cycles are also nested.*

Proof: (i) Let C be a cycle. Clearly $C' \subset C$. If $x \in C \setminus C'$ then $x \in \mathrm{int}(c)$ with c a cycle and $C \not\subset c$. Hence $\bigcup_{k\geq 0} f^k(B(x,\epsilon)) \subset c$ is not dense in C: $x \notin \mathrm{PEC}(C)$. Thus $\mathrm{PEC}(C) \subset C'$ and (i) is proved.

(ii) Let $C_1, C_2 \in \mathcal{C}$. If $C_1 = C_2$ or if the interiors of C_1 and C_2 are disjoint, (ii) is clear. We may assume that $C_2 \not\supset C_1$. Thus:

$$(*) \qquad \mathrm{int}(C_2) \subset X(C_1).$$

Let I_1 be an arbitrary connected component of C_1. Set $I_1' = \mathrm{conv}(I_1 \setminus X(C_1))$. Because of $(*)$, $I_1 \setminus X(C_1)$ has either points both to the left and to the right of $C_2 \cap I_1$ (case 1) or only on one side (case 2):

In case 1: $I_1' \supset C_2 \cap I_1$. Therefore, $C_1' \supset C_2$, so that the cycles, and the reduced cycles, are nested.

In case 2: $I_1' \cap \mathrm{int}(C_2) = \emptyset$. Therefore, $C_1' \cap \mathrm{int}(C_2) = \emptyset$ and $\mathrm{int}(C_1') \cap \mathrm{int}(C_2') = \emptyset$. ∎

FINITE MULTIPLICITY. We prove an "abstract" version of the announced Multiplicity Theorem 2.5 under purely topological hypothesis:

THEOREM 4.8: *Let $f\colon [0,1] \to [0,1]$ be a continuous map. If H is a constant strictly greater than $h_{\mathrm{loc}}(f)$ (assumed to be finite), then:*

The number of P.E.C. with topological entropy greater than H is finite.

In combination with the estimate on $h_{\mathrm{loc}}(f)$ of section 3, this will give the Multiplicity Theorem 2.5.

LEMMA 4.9: *Let $f\colon [0,1] \to [0,1]$ be a continuous map.*

For every $\epsilon > 0$ there exists $K < \infty$ such that, if C is a cycle of period n, then:

$$h_{\mathrm{top}}(f|C) \leq h_{\mathrm{loc}}(f,\epsilon) + K/n.$$

Proof: The topological entropy of C is bounded by $h_{\text{top}}(f|C,\epsilon)+h_{\text{loc}}(f,\epsilon)$. Let I_0 be an interval defined by C. Set $I_k=f^k(I_0)$. $I_0,\dots,I_{n-1}$ are the connected components of C. $f^n(I_0)=I_0$. Therefore $h_{\text{top}}(f|C,\epsilon)=h_{\text{top}}(f,I_0,\epsilon)\le \frac{1}{n}\log r(\epsilon,n,I_0)$. Hence:

$$h_{\text{top}}(f|C,\epsilon)\le\frac{1}{n}\left(\sum_{k=0}^{n-1}\log\left\lceil\frac{\text{diam}(I_k)}{\epsilon}\right\rceil\right).$$

Any single term of the sum is at most $\log\lceil\epsilon^{-1}\rceil$. For each k, the kth term is zero except if I_k has diameter greater than or equal to ϵ. The number of such I_k is less than ϵ^{-1}. Therefore:

$$h_{\text{top}}(f|C)\le h_{\text{loc}}(f,\epsilon)+(\epsilon^{-1}\log\lceil\epsilon^{-1}\rceil)/n. \quad ∎$$

Proof of the theorem: Let us call, during this proof, cycles C such that:

$$h_{\text{top}}(f|\operatorname{PEC}(C))\ge H,$$

distinguished cycles. Fix $\epsilon_0>0$ such that $h_{\text{loc}}(f,\epsilon_0)<H$.

Let us first make a simple remark. Let C be a cycle and assume that the diameter of $\operatorname{PEC}(C)\cap J$ is less than ϵ_0 for every connected component J of C. Then $\operatorname{PEC}(C)\subset\bigcup_J B_\infty(x_J,\epsilon_0)$ where J ranges over the (finitely many) connected components of C, and x_J is an arbitrary point of $\operatorname{PEC}(C)\cap J$. Therefore $h_{\text{top}}(f,\operatorname{PEC}(C))\le h_{\text{loc}}(f,\epsilon_0)<H$: C cannot be distinguished.

CLAIM: *One cannot have an "infinite tower":*

$$C_1\supsetneq C_2\supsetneq\cdots$$

of distinguished cycles.

Proof of the claim: Assume by contradiction that there exists such a tower $C_1\supsetneq C_2\cdots$. The previous lemma shows that distinguished cycles have their periods bounded by some constant. We can therefore assume that they have constant period n.

For every $i=1,2,\dots$, select J_i such that J_i is a connected component of C_i with:

$$\text{diam}(\operatorname{PEC}(C_i)\cap J_i)\ge\epsilon_0.$$

By taking a subsequence, we may assume that the J_i's are decreasing. Set $J = \bigcap_{i\geq 0} J_i$. It is a compact interval with diameter at least ϵ_0. Let L_i, resp. R_i, be the left, resp. right, connected component of $\overline{J_i \setminus J}$. Their diameters go to zero as $i \to \infty$. Remark that $\mathrm{PEC}(C_i) \cap J_i \subset L_i \cup R_i$. We may assume that $\mathrm{PEC}(C_i) \cap L_i$ is infinite for all $i \geq 1$.

The continuity of f implies that, for large i, $f^n(L_i)$ meet only one of L_i and R_i.

First case: $f^n(L_i) \subset L_i \cup J$. Hence, J_i being n-periodic, $\mathrm{orb}(L_i \cap \mathrm{PEC}(C_i)) \cap \mathrm{int}\, J_i \subset L_i$. As $\mathrm{PEC}(C_i)$ is topologically transitive, $\mathrm{orb}(L_i \cap \mathrm{PEC}(C_i)) \cap J_i = \mathrm{PEC}(C_i) \cap J_i \subset L_i$. But this contradicts $\mathrm{diam}(\mathrm{PEC}(C_i) \cap J_i) \geq \epsilon_0$.

So we are in the second case: $f^n(L_i) \subset R_i \cup J$. Hence, $\emptyset \neq f^n(\mathrm{PEC}(C_i) \cap L_i) \subset R_i$. If $f^n(R_i)$ did not meet L_i then no $f^k(R_i)$, $k \geq 0$ would, contradicting the transitivity of $\mathrm{PEC}(C_i)$. Hence $f^n(R_i) \subset L_i \cup J$. Therefore $\mathrm{PEC}(C_i) \cap L_i$ is f^{2n}-invariant. By the same remark as the one at the beginning of the proof, this implies that the (lim sup of the) topological entropy of $\mathrm{PEC}(C_i)$ is smaller than $h_{\mathrm{loc}}(f) < H$, a contradiction. The claim is proved. ∎

The claim ensures that:

(1) Every distinguished cycle contains some distinguished cycle which is **minimal** for inclusion.
(2) Every minimal distinguished cycle is contained in a finite number of distinguished cycles.

Therefore it is enough to show that the minimal distinguished cycles are in a finite number. As they are not nested, their reduced cycles are disjoint (see Lemma 4.7). But each one of them contains an interval with diameter at least $\epsilon_0 > 0$. This concludes the proof of the theorem. ∎

5. Global h-conjugation

We are going to prove the h-conjugation of the natural extension of $([0,1], f)$ with a T.M.C.

Let $f\colon [0,1] \to [0,1]$ be continuous. We may assume that $f(\{0,1\}) \subset \{0,1\}$. First we define the symbolic dynamics of $([0,1], f)$.

Definition 5.1: The **critical points** of f are the points of $[0,1]$ which have no neighbourhood on which f is strictly monotonic. The set of critical points is denoted by $C(f)$ and its backward orbit $\bigcup_{k\geq 0} f^{-k}C(f)$ by $C^{-}(f)$.

The **natural partition** is the collection P of the connected components of $[0,1]\setminus C(f)$. Any $x \in [0,1]\setminus C^{-}(f)$ defines a **real** (infinite) **itinerary**, $\Gamma(x) = A \in P^{\mathbb{N}}$ defined by:

$$f^k(x) \in A_k.$$

The set of real itineraries is:

$$\Sigma'_+(f,P) = \{A \in P^{\mathbb{N}}\colon \bigcap_{k\geq 0} f^{-k}(A_k) \neq \emptyset\}.$$

The symbolic dynamics is the closure of these:

$$\Sigma_+(f,P) = \overline{\Sigma'_+(f,P)} \subset P^{\mathbb{N}},$$

P being endowed with the *discrete topology.* The elements of $\Sigma_+(f,P)\setminus\Sigma'_+(f,P)$ are called **virtual itineraries**.

Each finite word $A_0\cdots A_n \in P^{n+1}$ defines a sub-interval:

$$[A_0\cdots A_n] = \bigcap_{k=0}^{n} f^{-k}(A_k).$$

Remarks: 1. If $A \in P$, then A and $f(A)$ are *open* subsets of $[0,1]$, $\overline{A}$ is disjoint from all other $B \in P$. In addition each restriction $f\colon \overline{A} \to \overline{f(A)}$ is a homeomorphism so that, for any set $S \subset \overline{A}$,

$$\partial f(S) = f(\partial(S)).$$

2. We note that $C(f)$ may well be equal to the entire $[0,1]$ interval. The class of piecewise monotonic maps studied by F. Hofbauer is characterized by $C(f)$ —and P— finite. Here $C(f)$ may be any compact, even uncountable, and P may be empty, finite, or countably infinite.

3. We notice that we choose to endow P with the discrete topology so as to add as few itineraries as possible.

For future reference, the **natural extension** of $\Sigma_+(f,P)$ is:

$$\Sigma(f,P) = \{(A_n)_{n\in\mathbb{Z}} \in P^{\mathbb{Z}}\colon \forall p \in \mathbb{Z} \quad (A_{p+n})_{n\geq 0} \in \Sigma_+(f,P)\}$$

under the action of the left shift σ.

Let us state the relationship between the original system $([0,1],f)$ and the symbolic system $(\Sigma_+(f,P),\sigma)$:

Definition 5.2: An interval $I \subset [0,1]$ with non-empty interior is said to be a **homterval** for f, if $f^k|I$ is a homeomorphism on its image, for every $k \geq 0$. We write $H(f)$ for the union of all the homtervals of f.

LEMMA 5.3: *There exists invariant subsets $\Sigma''_+(f,P) \subset \Sigma_+(f,P)$, $X'' \subset [0,1]$ such that the coding:*

$$\begin{array}{rcll} \Gamma\colon X'' & \longrightarrow & \Sigma''_+(f,P) & \\ x & \longmapsto & (A_k)_{k\geq 0} & \quad (f^k(x) \in A_k) \end{array}$$

is a measurable isomorphism.

Moreover $\Sigma''_+(f,P)$ is of full measure for any ergodic atomless invariant probability measure. The same hold for X'' with the additional condition that $C(f)$ is a null set for the measure.

More precisely, $X'' = [0,1] \setminus (H(f) \cup C^-(f))$.

It is well-known that, if f is $C^{1+\alpha}$ for some $\alpha > 0$, then any ergodic, invariant, probability measure of $([0,1], f)$ with positive entropy satisfies the condition above.

Proof: Set X'' as above. Remark that $H(f)$ is the countable union of the maximal homtervals (they have pairwise disjoint interiors) and that $f^{-1}(H(f)) \subset H(f) \cup C(f)$. If $H(f)$ were not a null set for some ergodic measure, the same would be true of some maximal homterval. But it is well known that points in a homterval are either wandering or tend to a periodic orbit. So the measure would have atoms: we may neglect $H(f)$. By invariance of the measure, we may neglect all of $[0,1] \setminus X''$.

Set $\Sigma''_+(f,P) = \Sigma_+(f,P) \setminus \Gamma(H(f) \setminus C^-(f))$. As an entire homterval defines one itinerary, $\Sigma'_+(f,P) \setminus \Sigma''_+(f,P)$ is countable. It remains to prove that $\Sigma_+(f,P) \setminus \Sigma'_+(f,P)$, the set of virtual itineraries, is countable:

If $A_0 A_1 \cdots$ is a virtual itinerary, then, as it is the limit of real itineraries, for every $n \geq 0$, $A_0 \cdots A_n$ is the beginning of some real itinerary and this implies $[A_0 \cdots A_n] \neq \emptyset$. But $\overline{[A_0 \cdots A_n]}$ is compact, so $\bigcap_{n\geq 0} \overline{[A_0 \cdots A_n]}$ is non-empty. Moreover,

$$\overline{[A_0 \cdots A_n]} \setminus [A_0 \cdots A_n] \subset \bigcup_{k=0}^{n} f^{-k} C(f)$$

so $\bigcap_{n\geq 0} \overline{[A_0 \cdots A_n]}$ contains a preimage by f^m $(m \geq 0)$ of some point $c \in C(f)$. Applying the homeomorphism $f^m|\overline{[A_0 \cdots A_m]}$, we get:

$$c \in \bigcap_{n\geq m} \overline{[A_m A_{m+1} \cdots A_n]}.$$

In particular $c \in \overline{A_m} \cap C(f) = \partial A_m = \{\min A_m, \max A_m\}$: c ranges in a countable set, as does (c, A_m). For $\epsilon > 0$ small enough, $B(c,\epsilon) \cap A_m \subset [A_m \cdots A_{m+k}]$. Hence A_{m+k} is the only element of P intersecting $f^k(B(c,\epsilon) \cap A_m)$ for every $\epsilon > 0$. So (c, A_m) determines the following symbols: $A_m A_{m+1} \cdots$. $\Sigma_+(f,P) \setminus \Sigma'_+(f,P)$ is indeed countable as obtained from a countable set by adding (at worst) arbitrary finite prefixes made up from the countable alphabet P. ∎

In general, $\Sigma_+(f,P)$ will not be a T.M.C. Nevertheless F. Hofbauer has shown that one can identify a part of the $\Sigma_+(f,P)$ with a T.M.C.

His proof relied on the peculiarities of the piecewise monotonic case but we shall show that, by modifying the construction, one can greatly generalize it: the result below is true for any symbolic system (i.e., a shift-invariant, closed subset of $A^{\mathbb{N}}$ with A countable and discrete).

Consider $\Sigma_*(f,P) = \{A_0 \cdots A_k \colon k \geq 0, A_0 \cdots \in \Sigma_+(f,P)\}$, the set of finite words. One easily sees that $A_0 \cdots A_k \in \Sigma_*(f,P)$ iff $[A_0 \cdots A_k] \not\subset C^-(f)$.

To build a Markov chain over $\Sigma(f,P)$, F. Hofbauer [10, eq. (2.1)] used as symbols of the enlarged "alphabet" the following sets:

$$\mathrm{fut}_X(A_{-n} \cdots A_0) = f^n[A_{-n} \cdots A_0], \qquad A_{-n} \cdots A_0 \in \Sigma_*(f,P)$$

which we call the **geometric futures** (see [15] for an exposition of Hofbauer's construction in a spirit similar to ours). The geometric future of a word clearly determines what symbols can come next, allowing the construction of a Markov chain. In the abstract setting which we are interested in, and also because it much clarifies the proof of the Isomorphism Theorem 5.7 below, we shall use instead the following.

Given a finite word $A_{-n} \cdots A_0 \in \Sigma_*(f,P)$, we say that $A_{-m} \cdots A_0 \in P^{m+1}$ is its **significant part**, if it is the smallest suffix necessary to compute the future, i.e., if m is minimal with the property that:

$$\mathrm{fut}_X(A_{-m} \cdots A_0) = \mathrm{fut}_X(A_{-n} \cdots A_0).$$

Of course a suffix is not a significant part in itself, but only with respect to a given finite word.

We write $\mathrm{sig}(A_{-n} \cdots A_0) = A_{-m} \cdots A_0$ and let $\tilde{P} = \{\mathrm{sig}(w) : w \in \Sigma_*(f,P)\}$ be our enlarged alphabet. We write $\tilde{\pi} : \tilde{P} \to P$ for the natural projection which sends $\mathrm{sig}(A_{-n} \cdots A_0) \mapsto A_0$.

We endow $\tilde{P}$ with the discrete topology and with the oriented graph structure defined by:

$$\mathrm{sig}(A_{-n}\cdots A_0) \to \mathrm{sig}(A_{-n}\cdots A_0 A_1)$$

for all $A_{-n}\cdots A_1 \in \Sigma_*(f,P)$. We call the graph $\tilde{P}$ **Hofbauer's diagram** for $\Sigma_+(f,P)$. We define:

$$\mathrm{fut}_X(\mathrm{sig}(A_{-n}\cdots A_0)) = \mathrm{fut}_X(A_{-n}\cdots A_0).$$

The oriented graph $\tilde{P}$ defines a two-sided topological Markov chain $\Sigma(\tilde{P}) \subset \tilde{P}^{\mathbb{Z}}$, which we call, after F. Hofbauer, its Markov extension. $\tilde{\pi}$ induces maps $\tilde{\pi} : \Sigma_+(\tilde{P}) \to \Sigma_+(f,P)$, $\tilde{\pi} : \Sigma(\tilde{P}) \to \Sigma(f,P)$. We shall check that they are well-defined.

LEMMA 5.4:

(i) *If $\beta_0 \to \cdots \to \beta_n$ is a finite path on $\tilde{P}$ then, for $0 \le k \le n$:*

$$\mathrm{fut}_X(\beta_k) = f^k(\mathrm{fut}_X(\beta_0) \cap [B_0\cdots B_k])$$

with $B_i = \tilde{\pi}(\beta_i)$ $(0 \le i \le n)$.

(ii) *Let $\alpha \in \tilde{P}$. $\tilde{\pi}$ restricted to the* **successors** *of α, i.e., the $\alpha' \in \tilde{P}$ such that $\alpha \to \alpha'$, is one-to-one.*

(iii) *If $B \in P^{\mathbb{N}}$ is such that $\mathrm{fut}_X(\alpha) \cap \bigcap_{k\ge 0}[B_0\cdots B_k] \ne \emptyset$, then there exists a unique infinite, one-sided path β on the graph $\tilde{P}$ such that $\tilde{\pi}(\beta) = B$ and $\beta_0 = \alpha$.*

COROLLARY 5.5: $\pi(\Sigma(\tilde{P})) \subset \Sigma(f,P)$.

Proof of the corollary: It is enough to prove that, for every $\beta \in \Sigma_+(P)$, $B = \tilde{\pi}(\beta)$ belongs to $\Sigma_+(f,P)$.

Let $k \ge 0$. (i) above implies that:

$$f^k([B_0\cdots B_k]) \supset \mathrm{fut}_X(\beta_k) = f^m([C_{-m}\cdots C_0])$$

with $C_{-m}\cdots C_0 \in \Sigma_*(f,P)$. Thus $[B_0\cdots B_k] \not\subset C^-(f)$. Hence there exists B'_i for $i > k$ such that $B_0\cdots B_k B'_{k+1}B'_{k+2}\cdots \in \Sigma'_+(f,P)$. But $k \ge 0$ was arbitrary and $\Sigma_+(f,P)$ is closed: $B \in \Sigma_+(f,P)$. ∎

Proof of the lemma: We prove (i) by induction. For $k = 0$ it is clear. Assume it for some $0 \le k < n$. By definition of $\tilde{P}$, there exists $C_{-m} \cdots C_0 C_1 \in \Sigma_*(f, P)$ such that $\beta_k = \mathrm{sig}(C_{-m} \cdots C_0)$ and $\beta_{k+1} = \mathrm{sig}(C_{-m} \cdots C_1)$. Hence:

$$\begin{aligned}\mathrm{fut}_X(\beta_{k+1}) &= f^{m+1}[C_{-m} \cdots C_1] = f^{m+1}[C_{-m} \cdots C_0] \cap C_1 \\ &= f(\mathrm{fut}_X(\beta_k)) \cap B_{k+1} \\ &= f(f^k(\mathrm{fut}_X(\beta_0) \cap [B_0 \cdots B_k])) \cap B_{k+1} \\ &= f^{k+1}(\mathrm{fut}_X(\beta_0) \cap [B_0 \cdots B_{k+1}])\end{aligned}$$

We have used the disjointness of the elements of P to see that $C_1 = B_{k+1}$. (i) is proved.

Let $\alpha \in \tilde{P}$ and α', α'' be two successors of α such that $\tilde{\pi}(\alpha') = \tilde{\pi}(\alpha'')$. There exists $A_{-n} \cdots A_1, B_{-m} \cdots B_1 \in \Sigma_*(f, P)$ such that:

$$\begin{gathered}\alpha = \mathrm{sig}(A_{-n} \cdots A_0) = \mathrm{sig}(B_{-m} \cdots B_0), \\ \alpha' = \mathrm{sig}(A_{-n} \cdots A_0 A_1), \quad \alpha'' = \mathrm{sig}(B_{-m} \cdots B_0 B_1), \\ A_1 = B_1.\end{gathered}$$

But:

$$\mathrm{fut}_X(A_{-n} \cdots A_0 A_1) = f(\mathrm{fut}_X(A_{-n} \cdots A_0)) \cap A_1$$

and similarly for B, so that the futures corresponding to the words $A_{-n} \cdots A_0$ and $B_{-m} \cdots B_0$ are equal. As the past did coincide before the addition of the symbol $A_1 = B_1$, they still do. So $\alpha' = \alpha''$. (ii) is proved.

To prove (iii), it is enough to set $\beta_k = \mathrm{sig}(A_{-n} \cdots A_{-1} B_0 \cdots B_k)$: by assumption $[A_{-n} \cdots B_k] \not\subset C^-(f)$, so this finite word does belong to $\Sigma_*(f, P)$. Uniqueness follows from part (ii) of the lemma. ∎

To state the isomorphism theorem, we need the:

Definition 5.6: Let $\Sigma_1(f, P) \subset \Sigma(f, P)$ stand for:

$$\{A \in \Sigma(f, P)\colon n \mapsto \mathrm{fut}_X(A_{-n} \cdots A_0) \text{ is not eventually constant}\}.$$

We define the **non-Markovian part** of $\Sigma(f, P)$ to be:

$$\Sigma_0(f, P) = \bigcup_{p \in \mathbb{Z}} \sigma^p \Sigma_1(f, P).$$

The following theorem generalizes F. Hofbauer's lemmas 5 and 6 of [10].

ISOMORPHISM THEOREM 5.7: *Let $\Sigma(\tilde{P})$ be the two-sided Markov extension defined above. Then the following diagram is commutative:*

$$\begin{array}{ccc} \Sigma(\tilde{P}) & \xrightarrow{\sigma} & \Sigma(\tilde{P}) \\ \tilde{\pi}\downarrow & & \downarrow\tilde{\pi} \\ \Sigma(f,P)\setminus\Sigma_0(f,P) & \xrightarrow{\sigma} & \Sigma(f,P)\setminus\Sigma_0(f,P) \end{array}$$

and this restriction of $\tilde{\pi}$ is a measurable isomorphism.

So for the natural extensions the Markov extension is (in this sense) *smaller* than the original system!

Proof:

STEP 1: We prove that the image of $\tilde{\pi}$ is included in $\Sigma(f,P)\setminus\Sigma_0(f,P)$ and that $\tilde{\pi}$ is one-to-one. Let $\alpha \in \Sigma(\tilde{P})$ and $A = \tilde{\pi}(\alpha) \in P^{\mathbb{Z}}$. For the first point, we remark that we have already seen that $A \in \Sigma(f,P)$, so it is enough, by σ-invariance of $\Sigma(\tilde{P})$, to show that $A \notin \Sigma_1(f,P)$.

Let $B_{-m}\cdots B_0 \in \Sigma_*(f,P)$ be such that:

$$\alpha_0 = \mathrm{sig}(B_{-m}\cdots B_0) \qquad \text{with } m \text{ minimal.}$$

Let $n \geq 0$. There exists a finite word $C_{-n-l}\cdots C_{-n} \in \Sigma_*(f,P)$ such that:

$$\alpha_{-n} = \mathrm{sig}(C_{-n-l}\cdots C_{-n}).$$

Necessarily, $C_{-n} = \tilde{\pi}(\alpha_{-n}) = A_{-n}$. By Lemma 5.4 (iii), for $i \leq n$,

$$\alpha_{-i} = \mathrm{sig}(C_{-n-l}\cdots C_{-n-1}A_{-n}\cdots A_{-i}).$$

Therefore:

(1) $\mathrm{sig}(B_{-m}\cdots B_0) = \mathrm{sig}(C_{-m-l}\cdots C_{-m-1}A_{-m}\cdots A_0)$ ($n = m$ and $i = 0$), implying that:

$$B_{-m}\cdots B_0 = A_{-m}\cdots A_0.$$

(2) Setting $n \geq m$ and $i = 0$, we get:

$$\mathrm{fut}_X(C_{-n-l}\cdots C_{-n-1}A_{-n}\cdots A_0) = \mathrm{fut}_X(B_{-m}\cdots B_0).$$

Compare the previous equation in (2) with $\text{fut}_X(A_{-n}\cdots A_0)$. It includes the left hand side and is included in the right hand side, according to (1). Therefore it is equal to their common value.

This shows that $\alpha_0 = \text{sig}(A_{-n}\cdots A_0)$ for all $n \geq m$. Therefore:

(1) $\text{fut}_X(A_{-n}\cdots A_0)$ is constant once $n \geq m$: $\alpha \notin \Sigma_1(f,P)$ as was to be shown;

(2) $\alpha_0 = \text{sig}(A_{-n}\cdots A_0)$ for n big enough, and likewise for α_p, $p \in \mathbb{Z}$. In particular, the map $\tilde{\pi}$: $\Sigma(\tilde{P}) \to P^{\mathbb{Z}}$ is one-to-one.

STEP 2: Let $A \in \Sigma(f,P) \setminus \Sigma_0(f,P)$: we have to prove that there exists some $\alpha \in \tilde{\Sigma}(f,P)$ such that $\tilde{\pi}(\alpha) = A$. Let $\alpha \in \tilde{P}^{\mathbb{Z}}$ be defined by:

$$\alpha_p = \lim_{n\to\infty} \text{sig}(A_{p-n}\cdots A_p) \qquad \text{for } p \in \mathbb{Z}.$$

As $A \notin \Sigma_0(f,P)$, for n big enough the equivalence class remains constant so α is well-defined in $\tilde{P}^{\mathbb{Z}}$. Clearly $\tilde{\pi}(\alpha) = A$.

For every $p \in \mathbb{Z}$, for n big enough,

$$\alpha_p = \text{sig}(A_{p-n}\cdots A_p),$$
$$\alpha_{p+1} = \text{sig}(A_{p-n}\cdots A_{p+1}),$$

so that:

$$\alpha_p \to \alpha_{p+1} \text{ is an arrow of } \tilde{P}$$

and $\alpha \in \Sigma(\tilde{P})$. So the map $\tilde{\pi} : \Sigma(\tilde{P}) \to \Sigma(f,P) \setminus \Sigma_0(f,P)$ is surjective and therefore bijective. It is clearly bi-measurable. ∎

Remarks: 1. As announced, this theorem does not assume *any property* for the underlying dynamical system $([0,1], f)$. It applies to the symbolic dynamics of any dynamic system, in fact, to any symbolic system.

2. For $\tilde{\pi}$ to be one-to-one in this general setting requires that $\tilde{P}$ was made of significant parts and not only made of the futures as in Hofbauer's original construction. To see this, one can consider —with obvious definitions— the case of the sofic system defined as the subshift of $\{0,1\}^{\mathbb{N}}$ excluding sequences containings blocks of 1's of odd-length.

3. Variants of Hofbauer's original construction are well-known in the theory of sofic systems defined by B. Weiss.

4. $\tilde{\pi}$ is continuous, but $\tilde{\pi}^{-1}$ is, in general, not even finitary (see [17]).

6. Entropy of the non-Markovian part

We are going to show that the dynamics through $C(f)$ give some control on the non-Markovian part of the symbolic dynamics introduced above.

THEOREM 6.1: *Let $\Sigma_0(f,P)$ be the non-Markovian part of the symbolic system defined by some continuous interval map.*

If μ is a σ-invariant probability measure concentrated on $\Sigma_0(f,P)$ then:

$$h(\mu) \leq h_{\text{top}}(f, C(f)),$$

i.e., $h_{\text{met}}(\Sigma_0(f,P)) \leq h_{\text{top}}(f, C(f))$.

The main, "geometric" property of $([0,1],f)$ which accounts for the above theorem is the following one:

$$\text{the cylinders } [A_0 \cdots A_n] \text{ are connected}, \qquad A_0 \cdots A_n \in \Sigma_*(f,P).$$

Another, more technical requirement will be to identify $\Sigma_0(f,P)$ with an invariant subset of the natural extension of $([0,1],f)$, so that we can exploit the (Bowen) topological entropy of the non-invariant subset $C(f)$, $h_{\text{top}}(f,C(f))$, which is defined with respect to the $[0,1]$-metric as the limit, when $\epsilon \to 0+$, of:

$$h_{\text{top}}(f,C,\epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log r(\epsilon,n,C).$$

On the interval, this identification goes through for any invariant, probability measure with no atomic ergodic component.

Let $A \in \Sigma(f,P)$. If $p < q$ are such that:

$$\text{fut}_X[A_p \cdots A_q] \subsetneq \text{fut}_X[A_{p+1} \cdots A_q]$$

then $[p,q]$ is called a **shadowing interval** for A. $A \in \Sigma_1(f,P)$ iff there exists shadowing intervals $[-n,0]$ with n arbitrarily large.

The connectedness of the cylinders implies the following fundamental property (which accounts for the terminology):

LEMMA 6.2: *Let $A \in \Sigma_0(f,P)$. If $[p,q]$ is a shadowing interval, then:*

$$[A_{p+1} \cdots A_q] \cap f(\partial A_p) \neq \emptyset.$$

Remarks: 1. This is the only property of the underlying dynamical system that we shall use in this section.

2. It says that $A_{p+1}\cdots A_q$ is the beginning of the itinerary of some point of $f(\partial A_p)$: so that, under suitable hypothesis, $A_pA_{p+1}\cdots$ represents some point $x\in[0,1]$ whose orbit stays close to the one of some point of $f(A_p)$ for the times $[p+1,q-T]$, with T some constant.

This explains why something like the above theorem should be true.

3. Finally, the inequality above is *sharp* in the sense that it is easy to build "abstract" examples, for instance invariant subsets S of full, bilateral shifts shadowed in an obvious sense by some compact subset of the corresponding unilateral shift. On the other hand, the statement with Bowen entropy substituted for metric entropy is false. So this is, in an essential way, a measure-theoretic statement.

A technical remark is in order. Notice that $\sigma^{-q-1}\Sigma_1(f,P)\subset\sigma^{-q}\Sigma_1(f,P)$. Indeed, if $[p,q+1]$ is a shadowing interval then so is $[p,q]$, because $[p,q+1]$ is a shadowing interval iff:

$$f(A_p)\not\supset[A_{p+1}\cdots A_{q+1}]$$

as $f\colon A_p\to f(A_p)$ is a bijection. But changing $q+1$ to q can only enlarge the right hand side.

Therefore, for any invariant, probability measure μ,

$$\sigma^{-q-1}\Sigma_1(f,P)=\sigma^{-q}\Sigma_1(f,P)\pmod{\mu}.$$

Hence the non-Markovian part can be written as:

$$\Sigma_0(f,P)=\bigcap_{p\in\mathbb{Z}}\sigma^p\Sigma_1(f,P)\cup N$$

with N of zero measure with respect to any probability invariant measure.

From now on, we forget about N and write $\Sigma_0(f,P)$ for the intersection: $A\in\Sigma_0(f,P)$ iff there exist shadowing intervals $[-n,m]$ with n,m arbitrarily large.

Proof of Lemma 6.2: The inequality of the futures in the definition of shadowing intervals implies, as f^{q-p} restricted to $[A_p\cdots A_q]$ is a bijection, that:

$$f(A_p)\not\supset[A_{p+1}\cdots A_q].$$

As $[A_{p+1}\cdots A_q]$ is connected and, by Remark 5.1, $\partial f(A_{-n-1})=f(\partial A_{-n-1})$, this is equivalent to:

$$f(\partial A_p)\cap[A_{p+1}\cdots A_q]\neq\emptyset. \qquad ∎$$

Proof of the theorem: Let μ be a σ-invariant probability measure concentrated on $\Sigma_0(f,P)$. As the metric entropy is affine, it is enough to consider the case with μ ergodic.

If μ has atoms then it is concentrated on a periodic orbit and the inequality to be proved is trivial. So we can also suppose μ atomless.

Γ: $[0,1] \setminus C^-(f) \to \Sigma_+(f,P)$ extends naturally to the natural extensions to a map $\bar\Gamma$: $\bar I \setminus \bar C(f) \to \Sigma(f,P)$ with $\bar C(f) = \bigcup_{p\in\mathbb{Z}} \bar f^p \bar\pi^{-1} C(f)$ (we define likewise $\bar H(f)$). Lemma 5.3 extends to the natural extensions and allows the identification (through $\bar\Gamma$) of μ with an ergodic and atomless $\bar f$-invariant probability measure on $\bar I$, which we shall also write μ. Set $I_0 = \bar\Gamma^{-1}(\Sigma_0(f,P)) \setminus \bar H(f)$. μ is concentrated on I_0.

We shall prove the desired bound by using the Katok formula for the metric entropy of a map on a compact metric space [12] (it is stated for a homeomorphism, but this is irrelevant):

$$h(\mu) = \lim_{\epsilon\to 0+} h(\mu,\epsilon) \quad \text{with } h(\mu,\epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log r(\epsilon,n,\mu),$$

$r(\epsilon,n,\mu)$ being the minimal cardinal of an (ϵ,n)-cover of μ, that is, the minimal number of (ϵ,n)-balls necessary to cover a set of μ-measure at least λ, with $\lambda\in(0,1)$, a constant, independent of n.

Fix $\epsilon>0$. We shall construct an (ϵ,n)-cover of μ. Set K_0 the cardinal of some finite $(\epsilon,1)$-cover of $\bar I$. Set $\alpha = \epsilon/5(\log K_0 + 1) < \epsilon$.

1. As $\bar f$ is continuous, there exists $\delta>0$ and $N_0<\infty$ such that, if $x,y\in\bar I$ and $|\bar\pi(\bar f^{-N_0}(x)) - \bar\pi(\bar f^{-N_0}(y))| < \delta$, then $d(x,y)<\epsilon$.

2. There exists $N_1<\infty$ such that, for all $n\geq N_1$, there exists R_n, an $(\delta/2,n)$-cover of $f(C(f))\subset[0,1]$ with cardinal less than $e^{(h_{top}(f,C(f))+\epsilon)n}$.

3. For μ-almost all $x\in\bar I$, $\bar\pi(x)\notin C^-(f)\cup H(f)$, so that, if $A=\Gamma(\bar\pi(x))$, $\bigcap_{k\geq 0} f^{-k}A_k = \{x\}$. But, for every n, $\bigcap_{k=0}^n f^{-k}A_k$ is an interval, therefore for μ-almost all $x\in\bar I$,

$$m(x) = \inf\{k\geq 1\colon \operatorname{diam}(P^k(\bar\pi(x))) < \delta/2\} < \infty.$$

One can select $I_1\subset I_0$ with $\mu(I_1) > 1-\alpha$ such that:

$$M_1 = \sup\{m(x)\colon x\in I_1\} < \infty.$$

Let $m_1 = \max(N_1, \alpha^{-1}M_1)$.

4. Let:

$$n(x) = \inf\{k \geq m_1\colon [-k, 0] \text{ is a shadowing interval for } \bar{\Gamma}(x)\}.$$

As $\mu(I_0) = 1$, $n(\cdot)$ is well-defined and finite almost everywhere. One can select $I_2 \subset I_1$ with $\mu(I_2) > 1 - 2\alpha$ such that:

$$M_2 = \sup\{n(x)\colon x \in I_2\} < \infty.$$

5. Applying Birkhoff's ergodic theorem, we find a measurable subset E of $\bar{I}$ with $\mu(E) > 1/2$ and an integer N_2 such that, if $x \in E$ and $n \geq N_2$,

$$\frac{1}{n}\operatorname{card}\{-N_0 \leq k < n - N_0\colon \bar{f}^k(x) \in I_2\} > 1 - 3\alpha.$$

Let $N_3 = \max(N_2, \alpha^{-1}M_2)$. For $x \in E$ and $n \geq N_3$, we cut $[-N_0, n - 1 - N_0]$ into shadowing intervals $[b_i, a_i]$: set $b_{-1} = n - N_0$ and, for $i \geq 0$,

(i) $a_i = \max\{k < b_{i-1} : n(\bar{f}^k(x)) \leq M_2\}$,
(ii) $b_i = a_i - n(a_i)$.

We notice that, for every $i \geq 0$, $\bar{\pi}(\bar{f}^{b_i+1}(x))$ is in $[A_{b_i+1} \cdots A_{a_i}]$ together with some point z_i of $f(C(f))$. We recall that there exists $z_i' \in R_{a_i - b_i}$ such that $|f^k(z_i) - f^k(z_i')| < \delta/2$, for $0 \leq k < a_i - b_i$. Therefore,

$$|f^k(\bar{\pi} \circ \bar{f}^{b_i}(x)) - f^k(z_i)| \leq \delta \qquad \text{for } 0 \leq k \leq a_i - b_i - M_1.$$

Let $r = \min\{i \geq 0\colon b_i < -N_0\} \leq n/m_1$. Set

$$A = \{-N_0 \leq k < n - N_0\colon \bar{f}^k(x) \in I_2\} \quad \text{and} \quad B = \bigcup_{i=0}^{r-1}[b_i, a_i - M_1] \cap A.$$

We have $\operatorname{card} A \geq (1 - 3\alpha)n$ and, as:

$$B = A \setminus \left(\bigcup_{i=0}^{r-1}[a_i - M_1 + 1, a_i] \cup [b_r, a_r]\right),$$

so,

$$\begin{aligned}\operatorname{card} B &\geq (1 - 3\alpha)n - rM_1 - M_2\\ &\geq (1 - 5\alpha)n.\end{aligned}$$

The knowledge of $z'_0, \dots, z'_{r-1}$ (at most $e^{(h_{\rm top}(f,C(f))+\epsilon)n}$ choices) and of the positions of intervals $[b_i, a_i]$, $0 \le i < r$ (at most

$$n/m_1 C_n^{2n/m_1} = \exp\bigl(n(h(2/m_1) + o(1))\bigr)$$

choices) determines $\bar{f}^k(x)$ with a precision ϵ for $k \in B + N_0$. But:

$$\operatorname{card}\left([0, n-1] \smallsetminus (B + N_0)\right) \le 5\alpha n \le \frac{\epsilon}{\log K_0 + 1} n.$$

Finally, within a precision of ϵ, there is at most $K_0^{5\alpha n}$ choices for the positions of $\bar{f}^k$ for $k \notin B + N_0$. As $K_0^{5\alpha n} \le e^{\epsilon n}$,

$$r(\epsilon, n, \mu) \le \exp\bigl(h(2/m_1) + (h_{\rm top}(f, C(f)) + \epsilon) + \epsilon + o(1)\bigr)n$$

with $h(t) = -t \log t - (1-t)\log(1-t)$ and $o(1)$ a function decreasing to zero as $n \to \infty$. First taking logarithms and dividing by n, then letting $n \to \infty$ and ϵ decrease to 0, proves the theorem. ∎

7. Proof of Theorems 2.5 and 2.9

We have already noted that the Multiplicity Theorem 2.5 is a direct corollary of 4.8 and 2.2, using the following result of B.M. Gurevič:

THEOREM 7.1 (B.M. Gurevič [9]): *Let G be a countable oriented graph. Assume G irreducible.*

1. *There exists at most one maximal measure for the topological Markov chain $(\Sigma(G), \sigma)$.*
2. *There exists exactly one maximal measure iff G is "positive-recurrent".*

The property of "positive-recurrence" has been introduced by D. Vere-Jones [20, 21]. We do not state its definition, as we will not use it — see Appendix A.

It only remains to prove the Structure Theorem 2.9.

Now we fix $K \subset [0,1]$ a P.E.C. with $h_{\rm top}(f|K) > \frac{1}{r}\log M(f)$. To apply the h-conjugation result of sections 5 and 6, we need the following estimate:

PROPOSITION 7.2: *Let f: $[0,1] \to [0,1]$ be C^r with $r \ge 1$ and Z be the zero set of f'. Then:*

$$h_{\rm top}(f, Z) \le \frac{1}{r} \log M(f).$$

Obviously $Z \supset C(f)$.

Proof: Fix $\epsilon > 0$. Write $h = h_{\text{top}}(f, Z)$. For $n \geq 1$, fix some maximal (ϵ, n)-separated subset of Z, i.e., $E_n \subset Z$ such that, if $x \in E_n$, $B_n(x, \epsilon) \cap E_n = \{x\}$,

$$E_n = \{x_1 < x_2 < \cdots < x_{N(n)}\}.$$

By definition of the Bowen entropy in terms of maximal separated sets [2], there exists arbitrarily large integers n, such that $N(n) \geq e^{(h-\epsilon)n}$. We consider only these n. Let $r = R + \alpha$ with R integer and $0 < \alpha \leq 1$. If there did not exist $k \in \{1, \ldots, N(n) - R + 1\}$ such that:

$$|x_{k+R-1} - x_k| \leq e^{-(h-2\epsilon)n}$$

then $x_N - x_1 \geq \left[\frac{N(n)}{R-1}\right] e^{-(h-2\epsilon)n} \to \infty$, which is absurd.

Therefore, for large n, there exists an interval containing at least R zeroes of f' and with diameter smaller than $e^{-(h-2\epsilon)n}$. Hence the diameter of $f([x_k, x_{k+R-1}])$ is bounded by $\text{Höl}_\alpha(f)\ \left[e^{-(h-2\epsilon)n}\right]^r$. But x_k, x_{k+1} are (ϵ, n)-separated so that there exists $0 \leq l < n$ such that:

$$\epsilon \leq |f^l(x_k) - f^l(x_{k+1})| \leq \text{Höl}_r(f) \left(e^{-(h-2\epsilon)n}\right)^r M(f)^{l-1}.$$

Taking logarithms and letting first $n \to \infty$, then $\epsilon \to 0$, we get the stated inequality. ∎

Thus the bound on $h_{\text{met}}(\sigma, \Sigma_0(f, P))$ established by Theorem 6.1 is enough to show that $\Pi = \bar{\Gamma}^{-1} \circ \bar{\pi}$ indeed defines a h-conjugation between (L, F), the natural extension of (K, f) and the shift on $\Pi^{-1}(L) \subset \Sigma(\tilde{P})$. It remains to show that the subset $\Pi^{-1}(L)$ is indeed defined by an irreducible part of $\tilde{P}$, that is:

$$\Pi^{-1}(L) = \Sigma(\tilde{I})\,\Delta\,N$$

with $\tilde{I}$ an irreducible part of $\tilde{P}$ (a subset $\tilde{P}$ defining an irreducible graph, maximal for inclusion) and N some set negligible for any $\mu \in \mathcal{M}^H(\Sigma(\tilde{P}))$, that is, invariant, ergodic, probability measure μ on $\Sigma(\tilde{P})$ with entropy $h(\mu) > \frac{1}{r} \log M(f)$.

Let $\tilde{\nu} \in \mathcal{M}^H(\Pi^{-1}(L))$. By ergodicity, $\tilde{\nu}$ is supported by $\Sigma(\tilde{I})$ with $\tilde{I}$ some irreducible part. But, $p \circ \Pi_* \tilde{\nu}$ is concentrated on $K \subset [0, 1]$ and, by h-conjugation (and easy properties of natural extensions), is ergodic and atomless. But we have:

PROPOSITION 7.3: *Let $f\colon [0,1] \to [0,1]$ be a continuous map and K be a transitive component with positive entropy. Then there exists at most one irreducible part $\tilde{I} \subset \tilde{P}$ such that $\mu(p \circ \tilde{\pi}(\Sigma_+(\tilde{I})) \cap K) > 0$ for some ergodic, atomless, invariant, probability measure.*

Assuming this proposition for the moment, we find that $\tilde{I}$ is independent of $\tilde{\nu}$. Therefore $N_+ = \Pi^{-1}(L) \setminus \Sigma(\tilde{I})$ has zero measure with respect to any $\tilde{\nu} \in \mathcal{M}^H(\Pi^{-1}(L))$, in fact any $\tilde{\nu} \in \mathcal{M}^H(\Sigma(\tilde{P}))$.

Π is continuous, therefore $\Pi(\Sigma(\tilde{I}))$ is topologically transitive. As it intersects the transitive component L on an infinite set, it must be contained in it: $\Pi^{-1}(L) \supset \Sigma(\tilde{I})$.

So $\Pi^{-1}(L) = \Sigma(\tilde{I}) \cup N_+$ and, setting $N = N_+$, the claim, and therefore the structure theorem, are proved. ∎

To prove the proposition above we first state a lemma of Hofbauer which subsists in our slightly different Markov extension:

LEMMA 7.4 (F. Hofbauer's lemma 4 of [11]): *Assume $\alpha, \beta \in \Sigma_+(\tilde{P})$ such that $\Pi(\alpha) = \Pi(\beta) \in [0,1]$ exists and does not belong to:*

$$H(f) \cup \bigcup_{k \geq 0} f^k C(f). \tag{7.1}$$

Then there exists $n < \infty$ such that:

$$\sigma^n(\alpha) = \sigma^n(\beta).$$

Proof: Let $A = \tilde{\pi}(\alpha)$. There exists $B_{-l} \cdots B_0 \in \Sigma_*(f,P)$, with $B_0 = A_0$, such that $\alpha_0 = \mathrm{sig}(B_{-l} \cdots B_0)$. By Lemma 5.4 (iii),

$$\alpha_k = \mathrm{sig}(B_{-m} \cdots B_{-1} A_0 \cdots A_k) \qquad \text{for } k \geq 0.$$

Also, as $f^{k+m}|[B_{-m} \cdots A_k]$ is a bijection,

$$\mathrm{fut}_X(B_{-m} \cdots B_{-1} A_0 \cdots A_k) = f^k(\mathrm{fut}_X(B_{-m} \cdots B_0) \cap [A_0 \cdots A_k]).$$

By assumption, the intervals $[A_0 \cdots A_k]$ reduce to $\{\Pi(\alpha)\}$ as $k \to \infty$ and $\Pi(\alpha)$ is not on the boundary of $\mathrm{fut}_X(B_{-m} \cdots B_0)$. Thus there exists some $k_1 \geq 0$, such that:

$$[A_0 \cdots A_k] \subset \mathrm{fut}_X(B_{-m} \cdots B_0),$$

in particular,

$$\alpha_k = \mathrm{sig}(A_0 \cdots A_k) \qquad \text{for } k \geq k_1.$$

The same is true with β instead of α for $k \geq$ some k_2. Therefore, $n = \max(k_1, k_2)$ has the stated property. ∎

Proof of the proposition: Let J be a periodic interval such that K is PEC(orb(J)). Let $\tilde{I}_1, \tilde{I}_2$ be irreducible parts of $\tilde{P}$ such that the related intersection with K has positive measure with respect to some ergodic, atomless, invariant probability measure. As we can avoid a set of zero measure, it is possible to select:

(i) $\alpha \in \Sigma_+(\tilde{I}_1)$ such that $x = \Pi(\alpha) \in K \cap \Omega(f)$ exists and is not in the backward orbit of the exceptional set (7.1) or in the boundary of orb(J).

(ii) $\gamma \in \Sigma_+(\tilde{I}_2)$ such that $z = \Pi(\gamma) \in K$ exists and is neither in the set defined by equation (7.1) or in the countable set Y defined by Lemma 4.5.

Fix $\epsilon > 0$ such that $B(x, \epsilon) \subset \mathrm{fut}_X(\alpha_0) \cap \mathrm{orb}(J)$. By Lemma 4.5,

$$\bigcup_{k \geq 0} f^k B(x, \epsilon) \supset \mathrm{orb}(J) \setminus Y \ni z$$

so that there exists $k \geq 0$ such that $f^k B(x, \epsilon) \ni z$. As $z \notin \bigcup_{l \geq 0} f^l(C(f))$, z is in fact in the interior of $f^k B(x, \epsilon)$. Set $y \in B(x, \epsilon)$ such that $f^k(y) = z$ and set $B = \Gamma(y)$. According to Lemma 5.4 (iii), α_0 and B together define an itinerary β on $\tilde{P}$ such that $\beta_0 = \alpha_0$ and $\Pi(\sigma^k \beta) = z$.

Now we apply the previous lemma to $\sigma^k(\beta)$ and γ: $\sigma^{k+n}(\beta) = \sigma^n(\gamma)$, for some $n \geq 0$. This implies that $\tilde{I}_1$ precedes $\tilde{I}_2$. By symmetry, the two irreducible parts precede each other, and so are equal. ∎

Appendix A. Counter-examples

TO THE EXISTENCE AND UNIQUENESS OF MAXIMAL ENTROPY TRANSITIVE COMPONENTS.

We build for arbitrarily large r, and any $\epsilon > 0$, C^r self-maps of the interval f_1, f_2 such that:

(i) f_1 has *no* transitive component of maximum entropy and:

$$h_{\mathrm{top}}(f_1) = \frac{1}{r} \log R(f_1).$$

(ii) f_2 has *infinitely many* transitive components of maximum entropy and:

$$\frac{1}{r}\log R(f_2) - \epsilon < h_{\rm top}(f_2) < \frac{1}{r}\log R(f_2)$$

with $\epsilon > 0$ arbitrarily small.

SKELETON. Let $r \geq 3$. Set $\lambda = (3e^{\epsilon})^r$. Pick a C^∞ piecewise monotonic map g: $[0,4] \to [0,4]$ respectively increasing, decreasing and increasing on the subintervals: $[0,1/2],[1/2,1],[1,4]$; and such that:

(i) $g(x) = \lambda x$ for $x \in [0, 3\lambda^{-1}]$.
(ii) $g([x_n, y_n]) = \lambda^{-n-\nu}[x_n, y_n]$ with $x_n = 1 + \frac{1}{n}$, $y_n = x_n + \frac{1}{n^2}$ for $n \geq 1$ ($\nu = 3$).
(iii) $g(4) \leq 1$.
(iv) $\max_{x\in[0,4]} |g'(x)| = \lambda$ (it is here that we need $\nu = 3$).
(v) $g^{(k)}(x_n) = g^{(k)}(y_n) = 0$ for all $k \geq 1$ and $n \geq 1$.

As $g([1,4]) \subset [0,1]$, the topological entropy of g is bounded by that of the 3-shift with subsequence 22 excluded: $h_{\rm top}(g) \leq \log(1+\sqrt{3}) < \log 3$.

EMBEDDING OF THE SUBSHIFTS. Select a C^∞ map s: $\mathbb{R} \to \mathbb{R}$ such that, for all $n \in \mathbb{Z}$,

(i) $s(x+2) = s(x)$ for all $x \in \mathbb{R}$.
(ii) $s(2n) = 0$, $s(2n+1) = 1$ for all $n \in \mathbb{Z}$ and s: $[n, n+1] \to [0,1]$ is a bijection, increasing if n is even, decreasing otherwise.
(iii) $s^{(k)}(n) = 0$ for all $k \geq 1$.
(iv) $\max_{x\in\mathbb{R}} |s'(x)| \leq 2$.

Let $M_n = 2E(m_n^{-1}3^n) + 1$, $n \geq 1$ with $m_n \geq 1$ a non-decreasing sequence of positive integers such that $\lim_{n\to\infty} \frac{1}{n}\log m_n = 0$. We define a map f: $[0,4] \to [0,4]$ by setting:

(i) $f(x) = g(x)$ if $x \notin \bigcup_{n\geq 1}[x_n, y_n]$.
(ii) $f(x) = \lambda^{-n-\nu}\left(x_n + (y_n - x_n)\cdot s\left(M_n \frac{x-x_n}{y_n-x_n}\right)\right)$ for $x \in J_n = [x_n, y_n]$.

We notice that, neglecting trivialities, each $(\mathrm{orb}(J_n), f)$ is a transitive component isomorphic to some obvious subshift of finite type. In particular,

$$h_{\rm top}(f|\,\mathrm{orb}(J_n)) = (n+\nu)^{-1}\log M_n \nearrow \log 3.$$

As each $\mathrm{orb}(J_n)$ is invariant, we see that any ergodic, invariant probability measure for f is concentrated either on some $\mathrm{orb}(J_n)$, or on the set

$[0,4] \setminus \bigcup_{n\geq 1} \mathrm{orb}(J_n)$. The metric entropy is thus bounded by $\log 3$ in the first case, and by $h_{\mathrm{top}}(g) \leq \log(1+\sqrt{3})$ in the second case. Therefore $h_{\mathrm{top}}(f) = \log 3$ and the maximal entropy transitive components are to be found exclusively among the $\mathrm{orb}(J_n), n \geq 1$.

Clearly f is C^∞ in the neighbourhood of any point $x \in [0,4] \setminus \{1\}$ and:

$$\sup_{x\in[0,4]\setminus\{1\}} |f'(x)| = (3e^\epsilon)^r.$$

If f is C^r in a neighbourhood of 1 then $f^{(k)}(1) = 0$ for all $0 \leq k \leq r$. So f is C^r on $[0,4]$ iff, when $x \to 0$,

(1) $\lim_{x\to 0} \frac{f^{(k)}(1+x)}{x} = 0$ for $k = 0, \ldots, r-1$,

(2) $f^{(r)}(1+x) \to 0$.

We notice that these conditions are already satisfied for $1+x$ restricted to $[0,4] \setminus \bigcup_{n\geq 1} J_n$. For $k \geq 0$ and $1+x \in \bigcup_{n\geq 0} J_n$, writing n for the only integer such that $1+x \in J_n$,

$$\begin{aligned} |f^{(k)}(1+x)| &\leq \lambda^{-n-\nu} M_n^k n^{2k-2} \\ &\leq \frac{n^{2k-2}}{m_n^k} \lambda^{-\nu}\, e^{-\epsilon n r}. \end{aligned}$$

We finally remark that, as 0 is a fixed point with $f'(x) = \lambda$, $R(f) = M(f)$.

Therefore we obtain the announced counter-examples to:

(i) the existence of a maximum entropy transitive component, if we set $\epsilon = 0$ and $m_n = n^2$;

(ii) the finiteness of their number, if we set $\epsilon > 0$ and $m_n = 1$.

Transitive, C^r Map Without Maximal Measure. Along the same lines as in the previous counter-examples, it is possible to define a C^r map, with $M(f) = (e^\epsilon 3)^r$, for any finite r such that the dynamics of f reduces to the one of the T.M.C. defined by the following graph G:

(1) G contains $\mathbb{N}$,

(2) there is one path, of length $3^{n+\nu}$, from n to all integers $m \geq n-1$, $m \neq n$,

(3) there are $3^{n-\nu}$ paths of length 3^n from n to n, for all $n \geq \nu$,

(4) there are no other arrows,

with ν some positive integer.

It is not difficult to see that $h_{\mathrm{met}}(\Sigma(G)) = \log 3$ (using Gurevič entropy [8]) and that we can remove any finite number of arrows without changing the entropy. Therefore G is not positive-recurrent (see I. Salama [18]) and it has no measure of maximal entropy by B.M. Gurevič's Theorem 7.1 above.

Appendix B. Semicontinuity of the metric entropy

We prove directly that local entropy bounds the defect in upper-semicontinuity of the metric entropy (Proposition 3.2).

Let μ be an invariant, probability measure on $([0,1], f)$. The metric entropy of μ is:

$$h(\mu) = \sup_P h(\mu, P) \tag{7.2}$$

where $h(\mu,P) = \inf_{n\geq 1} \frac{1}{n} H_\mu(P^{\vee n})$ with $P^{\vee n}$ the n-times iterated partition:

$$P^{\vee n} = P \vee f^{-1}P \vee \cdots \vee f^{-n+1}P = \{A_0 \cap f^{-1}A_1 \cap \cdots \cap f^{-n+1}A_{n-1} \neq \emptyset : A_i \in P\}$$

and $H_\mu(S) = -\sum_{A\in S} \mu(A) \log \mu(A)$.

Proof: First we are going to prove the following, apparently weaker, bound:

$$h_{usc}(f) \leq h_{\mathrm{loc}}(f) + \log 2. \tag{7.3}$$

That this implies the stated bound is standard: one considers f^q instead of f for $q \to \infty$ and remarks that

(i) $h_{f^q}(\nu) = q h_f(\nu)$,

(ii) $h_{\mathrm{loc}}(f^q) \leq q h_{\mathrm{loc}}(f)$ as $h_{\mathrm{loc}}(f^q, \delta) \leq q h_{\mathrm{loc}}(f, \epsilon)$ as soon as $|x - y| < \delta \implies |f^k(x) - f^k(y)| < \epsilon$ for $k = 0, \ldots, q-1$,

so that one may divide by q, replacing $\log 2$ by $\frac{\log 2}{q} \to 0$.

To prove (7.3), let $\epsilon > 0$ and μ be an invariant probability. Select a finite partition Q with diameter r so small that $h_{\mathrm{loc}}(f, r) \leq h_{\mathrm{loc}}(f) + \epsilon$ and such that $\mu(\bigcup_{A\in Q} \partial A) = 0$, so that μ is a point of upper-semicontinuity of $\nu \mapsto h(\nu, Q)$.

Clearly, for any finite partition P,

$$h(\mu, P) \leq h(\mu, Q) + \liminf_{n\to\infty} \frac{1}{n} H_\mu(P^{\vee n} | Q^{\vee n})$$

with $H_\mu(S|T) = H_\mu(S \vee T) - H_\mu(S)$ the conditional entropy of S knowing T.

We proceed as in the proof of the variational principle by M. Misiurewicz [13]. By regularity of μ and easy approximation results regarding the entropy of a finite partition, one can restrict the partition P in (7.2) to be a **compact partition** in the following sense:

$$P = \{K_1, \ldots, K_r, U\} \qquad \text{with } 1 \leq r < \infty \text{ and each } K_i \text{ compact.}$$

We claim that, for any compact partition P:

$$\limsup_{n\to\infty} \frac{1}{n} H_\nu(P^{\vee n}|Q^{\vee n}) \leq h_{\mathrm{loc}}(f) + \log 2 + 2\epsilon.$$

Once the claim will be proved for $\epsilon > 0$ arbitrarily small, (7.3) and therefore the proposition will follow. We turn to the claim:

It is immediate that:

$$H_\nu(P^{\vee n}|Q^{\vee n}) \leq \log \max_{E\in Q^{\vee n}} \mathrm{card}\{D \in P^{\vee n} : D \cap E \neq \emptyset\}.$$

Let $\delta = \frac{1}{2}\min\{d(K_i, K_j) : i \neq j\} > 0$ as P is a compact partition. Every set $E \in Q^{\vee n}$ is included in some (r, n)-ball. This Bowen ball can be covered using at most $e^{(h_{\mathrm{loc}}(f,r)+\epsilon)n} \leq e^{(h_{\mathrm{loc}}(f)+2\epsilon)n}$ (δ, n)-balls for large n. But if B is a (δ, n)-ball, $f^k(B)$ with $0 \leq k < n$ can meet at most one of the compact sets K_i (maybe also U).

Therefore the number of elements D of $P^{\vee n}$ that may meet E is bounded by:

$$2^n e^{(h_{\mathrm{loc}}(f)+2\epsilon)n},$$

proving the claim. ∎

References

[1] A. M. Blokh, *Decomposition of dynamical systems on an interval*, Russian Mathematical Surveys **38** (1983), 133–134.

[2] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Transactions of the American Mathematical Society **153** (1971), 401–414.

[3] R. Bowen, *Entropy-expansive maps*, Transactions of the American Mathematical Society **164** (1972), 323–333.

[4] J. Buzzi, *Number of equilibrium states of piecewise monotonics maps of the interval,* Proceedings of the American Mathematical Society **123** (1995), 2901–2907.

[5] J. Buzzi, *Intrinsic ergodicity of affine maps on* $[0,1]^d$, Monatshefte für Matematik (to appear).

[6] M. Denker, C. Grillenberg and K. Sigmund, *Ergodic theory on compact spaces,* Lecture Notes in Mathematics **527**, Springer-Verlag, Berlin, 1976.

[7] M. Gromov, *Entropy, homology and semi-algebraic geometry,* Séminaire Bourbaki **663**, 1985–1986.

[8] B. M. Gurevič, *Topological entropy of enumerable Markov chains,* Soviet Mathematics Doklady **10** (1969), 911–915.

[9] B. M. Gurevič, *Shift entropy and Markov measures in the path space of a denumerable graph,* Soviet Mathematics Doklady **11** (1970), 744–747.

[10] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy,* Israel Journal of Mathematics **34** (1979), 213–237; **38** (1981), 107–115.

[11] F. Hofbauer, *The structure of piecewise monotonic transformations,* Ergodic Theory and Dynamical Systems **1** (1981), 159–178.

[12] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms,* Publications Mathématiques de l'Institut des Hautes Études Scientifiques **51** (1980), 137–173.

[13] M. Misiurewicz, *A short proof of the variational principle for a* $\mathbb{Z}_+^n$ *action on a compact space,* Astérisque (International Conference on Dynamical Systems in Mathematical Physics), Vol. 40, 1976, pp. 147–158.

[14] S. E. Newhouse, *Continuity properties of the entropy,* Annals of Mathematics **129** (1989), 215–237.

[15] S. E. Newhouse, *On some results of F. Hofbauer on maps of the interval,* in *Dynamical Systems and Related Topics* (K. Shiraiwa, ed.), Proc. Nagoya 1990, World Scientific, Singapore, 1991, pp. 407–422.

[16] S. E. Newhouse and L.-S. Young, *Dynamics of certain skew products,* in *Geometric Dynamics,* Proc. Rio de Janeiro 1981, Lecture Notes in Mathematics **1007**, Springer-Verlag, Berlin, 1983.

[17] K. Petersen, *Ergodic Theory,* Cambridge University Press, 1983.

[18] I. A. Salama, *Topological entropy and classification of countable chains,* Ph.D. thesis, University of North Carolina, Chapel Hill, 1984.

[19] S. Smale, *Differentiable dynamics*, Bulletin of the American Mathematical Society **73** (1967), 97–116.

[20] D. Vere-Jones, *Geometric ergodicity in denumerable Markov chains*, The Quarterly Journal of Mathematics, Oxford **13** (1962), 7–28.

[21] D. Vere-Jones, *Ergodic properties of nonnegative matrices* I, Pacific Journal of Mathematics **22** (1967), 361–386.

[22] B. Weiss, *Intrinsically ergodic systems*, Bulletin of the American Mathematical Society **76** (1970), 1266–1269.

[23] Y. Yomdin, *Volume growth and entropy*, Israel Journal of Mathematics **57** (1987), 285–300.

[24] Y. Yomdin, *C^k-resolution of semi-algebraic mappings*, Israel Journal of Mathematics **57** (1987), 301–318.